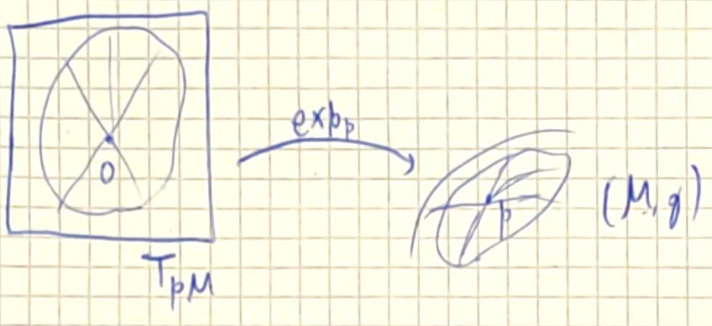
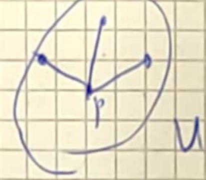
General Relativity $\boxed{\text{Week 5}}$



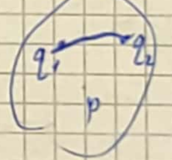
Last time: We used the fact that $\exp_p$ is a local diffeo around $v=0$ to define the <u>normal coordinates</u> around p

- A neighborhood U around p which is the diffeomorphic image through $\exp_p$ of a <u>star-shaped</u> domain in T_pM: <u>normal neighborhood</u>



← Every point in U is connected to p via a geodesic in U.

- <u>Convex neighborhood</u>: If $\forall q_1, q_2 \in U$, they are connected by a geodesic segment in U.



↗ Can be constructed as the intersection of normal neighborhoods of points
(The implicit function theorem provides <u>uniform</u> bounds for the coordinate size of these normal neighborhoods for points close enough to p)

<u>Normal coordinates</u> around p (on a normal neighborhood)

Choose orthonormal frame $\{e_0, \dots, e_n\}$ of T_pM

So $\forall v \in T_pM$: $v = v^a e_a$

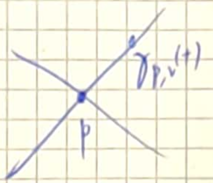
Define coordinates $(x^0, \dots, x^n)$ around p so that

$$x^a(\exp_p(v)) = v^a \iff x^a(q) = (\exp_p^{-1}(q))^a$$

In these coordinates:

If $\gamma_{p,v}(t) = \exp_p(vt)$

then $\gamma^k_{p,v}(t) = v^k t$.



<u>Prop</u>: Let $(x^0, \dots, x^n)$ be normal coordinates around $p \in (M,g)$.

Then $g_{\alpha\beta}(0) = \eta_{\alpha\beta}$, $\quad \partial_\alpha g_{\beta\gamma}(0) = 0$, $\quad \Gamma^\alpha_{\beta\gamma}(0) = 0$

(but, in general, $\partial^2_{\alpha\beta} g_{\gamma\delta}(0) \neq 0$ - this will correspond to the curvature tensor at p)

<u>Proof</u>: Note that the x^a-coordinate curve is $\exp_p(e_a t)$

So $\left.\frac{\partial}{\partial x^a}\right|_p = \left.\frac{d}{dt}\exp_p(e_a t)\right|_{t=0} = e_a$

Hence $g_{\alpha\beta}(0) = \left. g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)\right|_p = \eta_{\alpha\beta}$ (orthonormal basis).

And: $\forall v \in T_pM$:

$\gamma^k_{p,v}(t) = v^k t$ is a geodesic:

$$\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0 \Rightarrow \left.\Gamma^k_{ij}\right|_{\gamma(t)} v^i v^j = 0$$

So at $t=0$, This is true $\forall v$ and Γ^k_{ij} is symmetric in i,j $\Rightarrow \Gamma^k_{ij}(0) = 0$

And $\left.\partial_\alpha g_{\beta\gamma}\right|_p = \left.\Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma}\right|_p + \left.\Gamma^\lambda_{\alpha\gamma} g_{\beta\lambda}\right|_p = 0$ $\blacksquare$

Moreover, one can prove the analogue of the Gauss-lemma:
In normal coordinates around $p = (0, \dots, 0)$,
at every point $(x^0, \dots, x^n)$:

$$\boxed{g_{\alpha\beta} x^\beta = \eta_{\alpha\beta} x^\beta}$$

Differentiating 3 times and evaluating at $x=0$,
we get
$$\boxed{\partial^2_{\alpha\beta} g_{\gamma\delta}(0) + \partial^2_{\gamma\alpha} g_{\beta\delta}(0) + \partial^2_{\beta\gamma} g_{\alpha\delta}(0) = 0}$$

Local causal structure

Recall: $I^+_{(M,g)}(p) = \{q \in M : \exists \text{ timelike curve } p \to q\} \subseteq M$

$I^+_p = \{v \in T_pM : v \text{ fut. directed timelike}\} \subseteq T_pM$

(similarly for $J^\pm$)

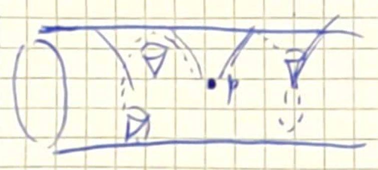
Locally: The causal structure around p is similar to that of Minkowski spacetime (i.e. locally causality is as in special relativity)

Prop: Let (M,g) be a spacetime and U_p be a normal neighborhood of p, such that $\exp_p : V_p \longrightarrow U_p$ is a diffeo
$V_p \subseteq T_pM$

Then
$$I^+_{U_p}(p) = \exp_p(I^+_p \cap V_p)$$

In particular, if $q \in I^+_{U_p}(p)$ then $\exists$ timelike geodesic from p to q.

Remark: • Note that, in general, $I^+_{U_p}(p) \subseteq I^+(p) \cap U_p$

Ex: 

$S^1 \times \mathbb{R}$: $I^+(p) = S^1 \times \mathbb{R}$

• Also: $J^+_{U_p}(p) \setminus I^+_{U_p}(p) = \exp_p\left((J^+_p \setminus I^+_p) \cap V_p\right)$, so

$J^+_{U_p}(p) \setminus I^+_{U_p}(p)$ is foliated by null-geodesics starting from p.

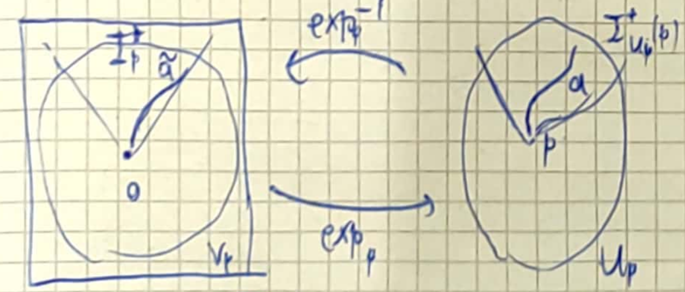
• $\overline{I^+_{U_p}(p)} = J^+_{U_p}(p)$

In general, we only have $\supseteq$, e.g. on $(\mathbb{R}^{1+1}, \eta)$ minus a point:

Proof:

• One direction is simple: $\exp_p(I^+_p \cap V_p) \subseteq I^+_{U_p}(p)$

this contains points connected to p through fut. directed timelike geodesics



• For the opposite: Let $\alpha: [0,1] \to U_p$ be a fut. directed timelike curve with $\alpha(0) = p$.

We need to show that $\tilde{\alpha} = \exp_p^{-1}(\alpha)$ stays in the cone I^+_p.

If $(x^0, \dots, x^n)$ are normal coordinates on U_p:

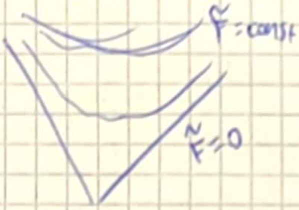
Set $F: U_p \to \mathbb{R}$, $F = -(x^0)^2 + (x^1)^2 + \dots + (x^n)^2$.

and $\tilde{F}: V_p \to \mathbb{R}$, $\tilde{F}(v) = F(\exp_p(v))$

Note: $\tilde{F}$ is the "Lorentzian length".

~~Staying inside~~ $\tilde{a}$ staying inside I_p^+

$$\iff \overbrace{F(a(t))}^{=F(t)} < 0 \quad \text{for } t > 0.$$



We calculate. If $a(t) = (x^0(t), \dots, x^n(t))$

$$F(t) = \eta_{\mu\nu} x^\mu(t) x^\nu(t)$$

$$\dot{F}(t) = 2\eta_{\mu\nu} \dot{x}^\mu(t) x^\nu(t)$$

$$\ddot{F}(t) = 2\eta_{\mu\nu} \ddot{x}^\mu(t) x^\nu(t) + 2\eta_{\mu\nu} \dot{x}^\mu(t) \dot{x}^\nu(t)$$

At $t=0$: $F(0)=0$, $\dot{F}(0)=0$, $\ddot{F}(0) < 0$ (since $\eta_{\mu\nu}(0) = g_{\mu\nu}(0)$)

So initially for $t \in (0, \varepsilon)$, $F(t) < 0$.

~~We need to~~ It suffices to show that

$\dot{F}(t) < 0 \quad \forall t$.

By Gauss lemma: $\dot{F}(t) = 2\eta_{\mu\nu} \dot{x}^\mu(t) x^\nu(t) = 2 g_{\mu\nu}(x(t)) \dot{x}^\mu(t) x^\nu(t)$

$$= 2 g(\dot{a}(t), \dot{\gamma}_{p, \exp_p^{-1}(a)}(t))$$

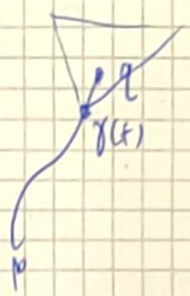
It is strictly negative as long as $\exp_p^{-1}(a) \in \vec{I}_p$!

So F can never become non-negative $\square$

Cor: If $S \subseteq M$, then $I^+(S)$ is open.

Pf: $I^+(S) = \bigcup_{p \in S} I^+(p)$

$I^+(p)$ is open since if $q \in I^+(p)$: If $\gamma : [0,1]$, $\gamma(0) = p$, $\gamma(1) = q$ is timelike future directed: If $t \in (0,1)$ is close enough to 1 so that q lies in a normal neighborhood U of $\gamma(t)$



$q \in I^+_U(\gamma(t))$

$= \exp_{\gamma(t)}(I^+_{\gamma(t)} \cap V)$ open

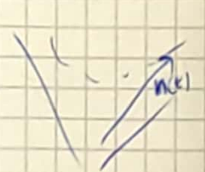
and $I^+_U(\gamma(t)) \subseteq I^+(p)$ $\square$

Local version of twin paradox

If $p \in (M,g)$ and U_p a normal neighborhood of p. If there exists a timelike curve γ from p to $q \in U_p$ then the radial geodesic $\tilde{\gamma}$ from p to q satisfies $l(\tilde{\gamma}) \geq l(\gamma)$, with equality iff $\gamma = \tilde{\gamma}$ up to reparametrization.

Pf: Parametrize γ by

$$\gamma(t) = \exp_p(r(t)\, \overbrace{n(t)}^{\text{"unit direction"}}) \quad \text{with } \langle n, n\rangle = -1 \quad (\text{so } \langle n, \dot n\rangle = 0)$$



Then: $$l(\gamma) = \int_0^1 \sqrt{-g(\dot\gamma,\dot\gamma)}\, dt = \int_0^1 \sqrt{-g_{\alpha\beta}\dot\gamma^\alpha \dot\gamma^\beta}\, dt$$

$$= \int_0^1 \sqrt{-g_{\alpha\beta}(\dot r \cdot n^\alpha + r \cdot \dot n^\alpha,\ \dot r n^\beta + r \cdot \dot n^\beta)}\, dt$$

normal coordinates $\gamma^\alpha(t) = r(t)\, n^\alpha(t)$

$$= \int_0^1 \sqrt{-g_{\alpha\beta} n^\alpha n^\beta \cdot (\dot r)^2 + r^2 \underbrace{g_{\alpha\beta}\dot n^\alpha \dot n^\beta}_{\geq 0}}\, dt$$ $\square$

Gauss lemma: Since n the radial vector: $g_{\alpha\beta}|_{\gamma(t)} n^\alpha = \eta_{\alpha\beta} n^\alpha$

So $\eta_{\alpha\beta} n^\alpha \dot n^\beta = 0 \Rightarrow \dot n$ is spacelike

Also: Corollary from the local theory:
If $q \in J^+(p) / I^+(p)$: p is connected
to q by a null geodesic

<u>Hierarchy of causality condition</u>: ← Global conditions!

Let (M, g) be a <u>spacetime</u> (i.e. time oriented)

- <u>Causal</u>: No closed causal curves

Not causal

↑ Very unstable condition!

- <u>Strongly causal</u>: No "almost" closed causal curves
  i.e. $\forall p \in M$, ~~open neighborhood of p~~:
  $\exists$ open neighborhood O of p: No causal curve intersects O more than once

↖ Causal but not strongly causal.

- <u>Stably causal</u>: Remain causal even under "small perturbations of the metric" (i.e. changes of the timecones)

<u>Def</u>: (M, g) is stably causal if $\exists f \in C^\infty(M)$ with
$\nabla^a f = g^{\alpha\beta}\partial_\beta f$ <u>Future directed</u> v.f.

↖ time function.

Note: $\nabla^a f$ remain timelike under small perturbations of the metric (open condition)
and along a timelike curve γ: $\dot{\gamma}(f) = \langle \dot{\gamma}, \nabla f \rangle < 0$.

So f strictly decreases along γ.